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LETTER TO THE EDITOR

Reflection from an asymmetric inhomogeneous medium

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Abstract. Reflection of waves from an asymmetric stratified medium is studied, whose refractive index obeys a power-law on both sides of a resonance point. Reflection is enhanced due to the asymmetry of the medium but the effect becomes weaker with increasing order of the discontinuity.

The effect of a discontinuous derivative of the refractive index upon the reflection of waves is generally expected to decrease with increasing order of the discontinuous derivative (Jacobsson 1966). It will be shown that this proposition also applies to the reflection at a resonance point, where the refractive index vanishes. Only the one-dimensional case in non-absorbing media (refractive index real) will be considered. Here a discontinuity of a derivative of the refractive index is not a sufficient condition for stronger reflection to occur. For this to be true, the profile must additionally be asymmetric with respect to the resonance point. In other words, the absolute value of some derivative must be discontinuous.

The waves are assumed to be governed by the equation $u''+\kappa(x)u=0$, where $u(x)$ is the amplitude and $\kappa(x)$ is a non-negative function (in case of electromagnetic waves proportional to the dielectric constant $\epsilon(x)$). Let the resonannce point be located at the origin $x=0$. For $\kappa(x)$ we assume a power law on both sides of that point,

$$\kappa(x):=\begin{cases} b_+x^p & x>0 \\ b_-(-x)^p & x<0 \end{cases}, \tag{1}$$

where b_+ and b_- are positive and p is a non-negative real number. In order to calculate the reflectivity for this particular medium we shall use the matrix method derived elsewhere (Robnik 1979, henceforth quoted as I), which differs from the approach by Försterling and Wüster (1950). Then the solution for u on the left and on the right (denoted by the indices $-/+$ respectively) can be written in terms of fundamental solutions $\psi_\pm$ as

$$u_\pm(x)=|W_\pm|^{-1/2}[A_\pm\psi_\pm(x)+B_\pm\psi_\pm^*(x)]. \tag{2}$$

However, on the right we have a transmitted wave only, hence $B_+=0$. To be determined is the complex reflection coefficient $R:=B_-/A_-$ which, according to I, equals $-G_{21}/G_{22}$, where G_{21} and G_{22} are the elements of the transmission matrix G,

$$G_{22}=-\mathrm{i}|W_+W_-|^{-1/2}(\psi_-^*\partial_x\psi_+-\psi_+\partial_x\psi_-^*),$$
$$G_{21}=-\mathrm{i}|W_+W_-|^{-1/2}(\psi_-\partial_x\psi_+-\psi_+\partial_x\psi_-). \tag{3}$$

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Our task is to find the fundamental solutions $\psi_\pm$. For $\kappa(x)$ from (1) they can be expressed in terms of Hankel functions of order $(p+2)^{-1}$, namely

$$\psi_+ = x^{1/2} H_\nu^{(1)}(2\nu b_+^{1/2} x^{1/2\nu}),$$
$$\psi_-{}^*(-x)^{1/2} H_\nu^{(2)}(2\nu b_-^{1/2}(-x)^{1/2\nu}), \tag{4}$$

with $\nu := (p+2)^{-1}$ (Jahnke *et al* 1960). The Wronskian is the same for the two cases, $W_+ = W_- = -2\mathrm{i}/\nu\pi$. It is imaginary negative, which ensures that the A-amplitude is associated with the forward energy flux (see also I). In a sufficiently small neighbourhood of the resonance point the ψ's are linear,

$$\psi_\sigma = [b_\sigma^{\nu/2}\nu^\nu/\Gamma(\nu+1)](\sigma + \mathrm{i}\cot(\nu\pi))x - \sigma\mathrm{i}\Gamma(\nu)/(\pi\nu^\nu b_\sigma^{\nu/2}), \tag{5}$$

where $\sigma = \pm$, and $\Gamma(\nu)$ is the gamma function. The values of the ψ's and their derivatives at the resonance point $x = 0$ can be taken from equation (5) and inserted into equations (3), whence

$$R = (\mathrm{i} + \rho\tan(\nu\pi))/(\mathrm{i} + \tan(\nu\pi)),$$
$$\rho = [(b_+/b_-)^\nu - 1]/[(b_+/b_-)^\nu + 1]. \tag{6}$$

The parameter ρ describing the effect of the asymmetry can also be rewritten in terms of the (discontinuous) pth derivative $\epsilon^{(p)}$ of the dielectric constant (if p is an integer),

$$\rho = \left(\frac{|\epsilon_+^{(p)}/\epsilon_-^{(p)}|^{1/(p+2)} - 1}{|\epsilon_+^{(p)}/\epsilon_-^{(p)}|^{1/(p+2)} + 1}\right). \tag{7}$$

From equation (6) follows the final expression for the reflectance $|R|^2$,

$$|R^2| = \cos^2(\nu\pi) + \rho^2\sin^2(\nu\pi). \tag{8}$$

The first term on the right refers to the reflection from a symmetric medium, while the effect of the asymmetry upon the reflection is described by the second term. To show this I wish to discuss a few special cases.

For a symmetric medium $b_+ = b_-$, hence $\rho = 0$, and one obtains

$$|R|^2 = \cos^2[\pi/(p+2)].$$

For even p this result agrees with the formula derived by Försterling and Wüster (1950). In particular, $p = 2$ yields the well known result $|R|^2 = \frac{1}{2}$.

If the medium is asymmetric, then the second term also contributes to the reflectivity and may be dominant for a small p. For example, if $p = 0$, definition (1) shows that we are dealing with two different semi-infinite homogeneous layers, and the elementary formula $|R|^2 = \rho^2$ follows. On the other hand, when p increases, whilst b_+/b_- remains constant, the importance of the discontinuity $b_+ \neq b_-$ becomes minor, because the parameters ρ and ν tend to zero. In the limit $p \to \infty$ the reflectance $|R|^2$ approaches unity independently of the asymmetry $b_+ \neq b_-$.

A final remark concerns the fact that a strict power-law profile of the medium seems unrealistic. Note, however, that the solutions $u_\pm$ from equation (2) with ψ's from (4) converge to the correct WKB solutions as $|x|$ increases. Secondly, if at some sufficiently large $|x|$ the power laws on both sides of the resonance point are cut off and matched continuously to homogeneous media, the profile is a realistic one. In this case the discontinuities of the derivative at the cut-off point would also contribute to the total reflectance. But, according to the result in I this contribution may be neglected if the value of the wave number at the cut-off point is sufficiently large.

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